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One loop photon-graviton mixing in an electromagnetic field: part 1

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Abstract: Photon-graviton mixing in an electromagnetic field is a process of potential interest for cosmology and astrophysics. At the tree level it has been studied by many authors. We consider the one-loop contribution to this amplitude involving a charged spin 0 or spin 1/2 particle in the loop and an arbitrary constant field. In the first part of this article, the worldline formalism is used to obtain a compact two-parameter integral representation for this amplitude, valid for arbitrary photon energies and background field strengths. The calculation is manifestly covariant througout.

Keywords: Cosmology of Theories beyond the SM, Electromagnetic Processes and Properties, Models of Quantum Gravity.

Contents

1. Introduction

In the presence of an external electromagnetic field, many quantum processes exist which are forbidden in vacuum (see, e.g., [1, 2]). In particular, transitions between bosons of different spin become possible [3, 4, 5]. One such process which has been well-studied is the axion-photon mixing in a magnetic field [5, 6, 7]. Similarly, also photon-graviton mixing is possible in an external field [3, 4, 5]. The corresponding tree level amplitude is contained in the coupling $h_{\mu\nu}T^{\mu\nu}$ of the graviton $h_{\mu\nu}$ to the energy-momentum tensor $T^{\mu\nu}$ of the electromagnetic field,

$$T^{\mu\nu} = F^{\mu\alpha}F^{\nu}_{\ \alpha} - \frac{1}{4}F_{\alpha\beta}F^{\alpha\beta}\eta^{\mu\nu}\,. \tag{1.1}$$

Namely, when taking $F^{\mu\nu} = F^{\mu\nu}_{\rm ext} + f^{\mu\nu}$ with $F^{\mu\nu}_{\rm ext}$ the external field strength tensor and $f^{\mu\nu}$ the photon field, $h_{\mu\nu}T^{\mu\nu}$ yields the trilinear term

$$h_{\mu\nu}\Bigl(F^{\mu\alpha}_{\rm ext}f^{\nu}_{\ \alpha} + f^{\mu}_{\ \alpha}F^{\nu\alpha}_{\rm ext}\Bigr) - \frac{1}{2}h^{\mu}_{\mu}F^{\alpha\beta}_{\rm ext}f_{\alpha\beta}\,. \tag{1.2}$$

Due to the smallness of the gravitational coupling constant κ this process has attracted less attention than the axion-photon coupling. Nevertheless, its relevance for astrophysics has been scrutinized by a number of authors (see [8] for a discussion of possible laboratory experiments). In [5] photon-graviton conversion near a pulsar was studied but the transition rate was found to be very small. In [9, 10] it has been suggested that the same conversion due to a primordial magnetic field could be responsible for the observed anisotropy of the

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cosmic microwave background. However, [11] find that its effect becomes negligible for standard cosmological magnetic fields if plasma effects are taken into account. Renewed interest in this amplitude has been generated by the recent models with large extra dimensions [12]. These models contain additional massive Kaluza-Klein gravitons which might lead to an enhancement of the photon-graviton conversion effect. See [13] for a discussion of possible observable effects in astrophysics as well as in the laboratory. In the same context also graviton-photon conversion on spin 0 and 1/2 particles was considered [14].

To our knowledge, the photon-graviton amplitude has so far been considered only at tree level. In the present paper, we extend its study to include the one-loop corrections due to virtual spin 0 or 1/2 particles. Contrary to the tree level case, at one loop these amplitudes depend nontrivially on both the photon energy and the background field strength. Thus it is conceivable that, for some region in this two-parameter space, the one-loop amplitudes might be comparable with or dominating over the tree level ones. In the first part of this article, we use the 'string-inspired' worldline formalism to obtain compact integral representations for these amplitudes. Their numerical study will be undertaken in the second part.

Our calculation provides also a first example for the application of the string-inspired technique to mixed photon-graviton amplitudes. In recent years, methods derived from [15] or inspired by [16] string theory have been extensively used for the calculation of on-shell gluon [17] and graviton [18, 19] amplitudes, as well as for QED amplitudes in a constant field [20]–[27]. More recently, in [28] also the contribution to the graviton vacuum polarization involving a scalar loop was obtained. The treatment of gravitational backgrounds in the wordline formalism involves a number of mathematical subtleties which have been clarified only recently [29]–[35].

Phenomenologically, the photon-graviton amplitude is primarily of interest in the magnetic field case. However, in the formalism used here it makes technically no essential difference whether one calculates an amplitude in a constant magnetic or in a general constant field. Thus we will keep the electric component.

The organization of the paper is simple: In section 2 we present the general formalism for calculating one loop amplitudes involving either a scalar or spinor loop and any number of photons and gravitons, in vacuum or in a constant external field. Section 3 contains the scalar loop calculation, section 4 the spinor loop one. Our conclusions are given in 5. In the appendix we verify that the results obey the relevant gauge and gravitational Ward identities.

2. Mixed electromagnetic — gravitational amplitudes in the worldline formalism

The application of the worldline formalism to flat space calculations has been described in detail in the review [36]. However, the generalization to processes involving curved backgrounds, or gravitons, is less familiar. Therefore, after a brief description of the worldline formalism adapted to the case of a constant electromagnetic background, we will describe certain subtleties arising in this formalism with the coupling to gravity.

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Let us first consider the case of a scalar particle coupled to electromagnetism and gravity. We use here euclidean conventions. The QFT for this scalar particle is described by a complex scalar field ϕ with action

$$S[\phi,\phi^*;g,A]=\int d^Dx\sqrt{g}\Big[g^{\mu\nu}(\partial_\mu-ieA_\mu)\phi^*(\partial_\nu+ieA_\nu)\phi+(m^2+\xi R)\phi^*\phi\Big] \tag{2.1}$$

where ξ parameterizes an additional non-minimal coupling to the scalar curvature.[1] The corresponding one-loop effective action is formally given by[2]

$$\begin{aligned} e^{-\Gamma[g,A]} &= \int \mathcal{D}\phi\mathcal{D}\phi^*\, e^{-S[\phi,\phi^*;g,A]} \\ \Gamma[g,A] &= -\log \mathrm{Det}^{-1}(-\Box_A+\xi R+m^2)=\mathrm{Tr}\log(-\Box_A+\xi R+m^2) \end{aligned} \tag{2.2}$$

where $\Box_A$ is the gauge and gravitationally covariant laplacian for scalar fields obtained from (2.1). This effective action can be represented in the worldline formalism by

$$\Gamma[g,A]=-\int_0^\infty\frac{dT}{T}\int_{PBC}\mathcal{D}x\,e^{-S[x;g,A]} \tag{2.3}$$

with the worldline action

$$S[x;g,A]=\int_0^T d\tau\left(\frac{1}{4}g_{\mu\nu}(x)\dot{x}^\mu\dot{x}^\nu+ieA_\mu(x)\dot{x}^\mu+\xi R(x)+m^2\right). \tag{2.4}$$

This worldline representation contains a standard integral over the proper time T, and a quantum mechanical path integral over the particle coordinates $x^\mu(\tau)$. Due to the trace in (2.2) the path integral is to be taken with periodic boundary conditions $x^\mu(0)=x^\mu(T)$. Thus it corresponds to an integral over closed loops in spacetime. In the worldline or 'string-inspired' formalism, the path integral (2.3) is manipulated into gaussian form and then evaluated using appropriate worldline correlators. In an amplitude calculation, the external fields are specialized to plane waves. Then each external leg is represented by a vertex operator. For the electromagnetic field this is the photon vertex operator

$$V^A_{\rm scal}[k,\varepsilon]=\varepsilon_\alpha\int_0^T d\tau\,\dot{x}^\alpha(\tau)\mathrm{e}^{ik\cdot x(\tau)} \tag{2.5}$$

with an associated coupling constant $-ie$. The graviton vertex operator will be derived below. For the calculation of the path integral, one first splits off the loop average position:

$$x^\mu(\tau)=x_0^\mu+y^\mu(\tau)\,, \tag{2.6}$$

$$x_0^\mu\equiv\frac{1}{T}\int_0^T d\tau\,x^\mu(\tau)\,. \tag{2.7}$$

[1]The value $\xi=0$ is the minimal coupling, while the value $\xi=\frac{D-2}{4(D-1)}$ gives a conformally invariant coupling in the massless case.

[2]Our present definition of the effective action differs by a sign from the conventions in [36]; however, there is no difference at the amplitude level.

The path integral then factors into $\int Dx(\tau) = \int d^D x_0 \int Dy(\tau)$. The integral over x_0 just produces the global delta function for energy-momentum conservation. The reduced path integral $\int Dy(\tau)$ is gaussian and can be calculated using the Wick contraction rule [16]

$$\langle y^\mu(\tau_1) y^\nu(\tau_2)\rangle = -\delta^{\mu\nu}\, G_B(\tau_1,\tau_2)\,. \tag{2.8}$$

Here G_B is the 'worldline Green's function'

$$\begin{aligned} G_B(\tau_1,\tau_2) &= |\,\tau_1-\tau_2\,| - \frac{(\tau_1-\tau_2)^2}{T}\,,\\ \dot G_B(\tau_1,\tau_2) &= \mathrm{sign}(\tau_1-\tau_2) - 2\frac{(\tau_1-\tau_2)}{T}\,,\\ \ddot G_B(\tau_1,\tau_2) &= 2\delta(\tau_1-\tau_2) - \frac{2}{T}\,. \end{aligned} \tag{2.9}$$

Here and in the following we will often abbreviate $G_{B12} \equiv G_B(\tau_1,\tau_2)$ etc., and a 'dot' generally refers to a derivative in the first variable.

The inclusion of a constant electromagnetic background field $F_{\mu\nu}$ can be achieved easily [20, 23] using Fock-Schwinger gauge centered at the loop average position x_0. In this gauge

$$A_\mu(x) = \frac{1}{2} y^\nu F_{\nu\mu} \tag{2.10}$$

so that the presence of the background field produces only an additional term

$$\Delta L = \frac{1}{2} i e y^\mu F_{\mu\nu} \dot y^\nu \tag{2.11}$$

to the worldline lagrangian in (2.4). Since this terms involves y only quadratically it can be taken into account by an appropriate change of the worldline correlators. Namely, instead of (2.8), (2.9) one finds

$$\langle y^\mu(\tau_1) y^\nu(\tau_2)\rangle = -\mathcal{G}_B^{\mu\nu}(\tau_1,\tau_2) \tag{2.12}$$

with a field dependent worldline correlator [37, 23]

$$\begin{aligned} \mathcal{G}_B(\tau_1,\tau_2) &= \frac{T}{2(\mathcal{Z})^2}\left(\frac{\mathcal{Z}}{\sin(\mathcal{Z})}\mathrm{e}^{-i\mathcal{Z}\dot G_{B12}} + i\mathcal{Z}\dot G_{B12} - 1\right),\\ \dot{\mathcal{G}}_B(\tau_1,\tau_2) &= \frac{i}{\mathcal{Z}}\left(\frac{\mathcal{Z}}{\sin(\mathcal{Z})}\mathrm{e}^{-i\mathcal{Z}\dot G_{B12}} - 1\right),\\ \ddot{\mathcal{G}}_B(\tau_1,\tau_2) &= 2\delta_{12} - \frac{2}{T}\frac{\mathcal{Z}}{\sin(\mathcal{Z})}\mathrm{e}^{-i\mathcal{Z}\dot G_{B12}}\,. \end{aligned} \tag{2.13}$$

where $\mathcal{Z}_{\mu\nu} \equiv eTF_{\mu\nu}$. These expressions should be understood as power series in the Lorentz matrix $\mathcal{Z}$; more explicit expressions are given below. Note the symmetry properties

$$\mathcal{G}_{B12} = \mathcal{G}_{B21}^T\,, \qquad \dot{\mathcal{G}}_{B12} = -\dot{\mathcal{G}}_{B21}^T\,, \qquad \ddot{\mathcal{G}}_{B12} = \ddot{\mathcal{G}}_{B21}^T\,. \tag{2.14}$$

Contrary to the vacuum case, in a constant field background the worldline correlators have non-vanishing coincidence limits,

$$\begin{aligned}\mathcal{G}_B(\tau,\tau) &= \frac{T}{2(\mathcal{Z})^2}\Big(\mathcal{Z}\cot(\mathcal{Z})-1\Big)\,,\\ \dot{\mathcal{G}}_B(\tau,\tau) &= i\cot(\mathcal{Z})-\frac{i}{\mathcal{Z}}\,.\end{aligned} \tag{2.15}$$

In the following a 'bar' on a quantity denotes the subtraction of its coincidence limit, e.g.,

$$\overline{\mathcal{G}}_{B12} \equiv \mathcal{G}_{B12} - \mathcal{G}_{B11}\,. \tag{2.16}$$

Note that coincidence limits of worldline correlators are always constant due to their translational invariance.

The path integral determinant also becomes field dependent. In flat space [20]

$$\int Dy(\tau)\exp\left[-\int_0^T d\tau\left(\frac{1}{4}\dot{x}^2+\frac{1}{2}ie\,x^\mu F_{\mu\nu}\dot{x}^\nu\right)\right] = (4\pi T)^{-\frac{D}{2}}\det{}^{-\frac{1}{2}}\left[\frac{\sin(\mathcal{Z})}{\mathcal{Z}}\right]. \tag{2.17}$$

The path integral representing a spin $\frac{1}{2}$ particle in an electromagnetic field differs from the spin 0 case (2.3) above by a global factor of $-\frac{1}{2}$, and by an additional Grassmann path integral representing the spin,

$$\int_{ABC}\mathcal{D}\psi\exp\left[-\int_0^T d\tau\left(\frac{1}{2}g_{\mu\nu}(x)\psi^\mu\Big(\dot{\psi}^\nu+\dot{x}^\alpha\Gamma^\nu_{\alpha\beta}(x)\psi^\beta\Big)-ie\psi^\mu F_{\mu\nu}(x)\psi^\nu\right)\right]. \tag{2.18}$$

Here the path integral is over antiperiodic Grassmann functions, $\psi^\mu(T)=-\psi^\mu(0)$. In the vacuum case, the appropriate worldline correlator is

$$\langle\psi(\tau_1)\psi(\tau_2)\rangle = \frac{1}{2}G_F(\tau_1,\tau_2) \equiv \frac{1}{2}\mathrm{sign}(\tau_1-\tau_2)\,. \tag{2.19}$$

For a constant background field this correlator turns into

$$\langle\psi^\mu(\tau_1)\psi^\nu(\tau_2)\rangle = \frac{1}{2}\mathcal{G}_F^{\mu\nu}(\tau_1,\tau_2) \tag{2.20}$$

where

$$\begin{aligned}\mathcal{G}_F(\tau_1,\tau_2) &= G_{F12}\frac{\mathrm{e}^{-i\mathcal{Z}\dot{G}_{B12}}}{\cos(\mathcal{Z})},\\ \dot{\mathcal{G}}_F(\tau_1,\tau_2) &= \dot{G}_{F12}+G_{F12}\frac{2i\mathcal{Z}}{T\cos(\mathcal{Z})}\mathrm{e}^{-i\mathcal{Z}\dot{G}_{B12}}\,.\end{aligned} \tag{2.21}$$

Its symmetry properties are

$$\mathcal{G}_F(\tau_1,\tau_2) = -\mathcal{G}_F^T(\tau_2,\tau_1)\,,\qquad \dot{\mathcal{G}}_F(\tau_1,\tau_2) = \dot{\mathcal{G}}_F^T(\tau_2,\tau_1)\,. \tag{2.22}$$

The coincidence limits are

$$\mathcal{G}_{F11} = -i\tan(\mathcal{Z})\,,\qquad \dot{\mathcal{G}}_{F11} = 2\delta_{11}+2eF\tan(\mathcal{Z})\,. \tag{2.23}$$

The free path integral in a constant $F_{\mu\nu}$ background in D dimensions is normalized as

$$\int \mathcal{D}\psi \exp\Big[-\int_0^T d\tau \Big(\frac{1}{2}\psi \cdot \dot{\psi} - ie\psi^\mu F_{\mu\nu}\psi^\nu\Big)\Big] = 2^{\frac{D}{2}} \det{}^{\frac{1}{2}}\Big[\cos(\mathcal{Z})\Big]. \tag{2.24}$$

The photon vertex operator (2.5) acquires an additional Grassmann piece,

$$V^A_{\rm spin}[k,\varepsilon] = \varepsilon_\alpha \int_0^T d\tau \Big[\dot{x}^\alpha(\tau) + 2i\psi^\alpha(\tau)\psi(\tau)\cdot k\Big] e^{ik\cdot x(\tau)}. \tag{2.25}$$

In the flat space case, the naive gaussian path integration gives unambiguous and well-defined parameter integral representations. To the contrary, when the coupling to gravity is introduced more precise definitions of the worldline regularizations are required to define the path integral properly. Let us describe these issues briefly. First of all it is convenient to exponentiate the nontrivial path integral measure in a regularization independent way by using ghost fields [29, 30]. The covariant measure in (2.3) is of the form

$$\mathcal{D}x = Dx \prod_{0\le\tau<T} \sqrt{\det g_{\mu\nu}(x(\tau))} \tag{2.26}$$

where $Dx = \prod_\tau d^D x(\tau)$ is the standard translationally invariant measure. It can be represented more conveniently by introducing commuting a^μ and anticommuting b^μ, c^μ ghosts with periodic boundary conditions

$$\mathcal{D}x = Dx \prod_{0\le\tau<1} \sqrt{\det g_{\mu\nu}(x(\tau))} = Dx \int_{PBC} DaDbDc\, e^{-S_{gh}[x,a,b,c]} \tag{2.27}$$

where the ghost action is given by

$$S_{gh}[x,a,b,c] = \int_0^T d\tau\, \frac{1}{4} g_{\mu\nu}(x)(a^\mu a^\nu + b^\mu c^\nu). \tag{2.28}$$

The extra vertices arising from the ghost action guarantee that the final result will be finite (see, e.g., [38]).

For the perturbative calculation of graviton amplitudes around flat space, one next linearizes the metric,

$$g_{\mu\nu}(x) = \delta_{\mu\nu} + \kappa h_{\mu\nu}(x) \tag{2.29}$$

and then specializes $h_{\mu\nu}(x)$ to plane wave form,

$$h_{\mu\nu}(x) = \varepsilon_{\mu\nu} e^{ik\cdot x}. \tag{2.30}$$

This leads to the following vertex operator for the graviton coupled to the loop scalar

$$V^h_{\rm scal}[k,\varepsilon] = \varepsilon_{\mu\nu} \int_0^T d\tau \Big[\dot{x}^\mu(\tau)\dot{x}^\nu(\tau) + a^\mu(\tau)a^\nu(\tau) + b^\mu(\tau)c^\nu(\tau) + 4\bar{\xi}(\delta^{\mu\nu}k^2 - k^\mu k^\nu)\Big] e^{ik\cdot x(\tau)} \tag{2.31}$$

with an associated coupling constant factor of $-\kappa/4$. The Wick contraction rules for the ghosts are

$$\begin{aligned}\langle a^\mu(\tau_1)a^\nu(\tau_2)\rangle &= 2\delta(\tau_1-\tau_2)\delta^{\mu\nu}\,,\\ \langle b^\mu(\tau_1)c^\nu(\tau_2)\rangle &= -4\delta(\tau_1-\tau_2)\delta^{\mu\nu}\,.\end{aligned} \tag{2.32}$$

Similarly, for the fermion loop case one finds a graviton vertex operator

$$\begin{aligned}V^h_{\rm spin}[k,\varepsilon] = \varepsilon_{\mu\nu}\int_0^T d\tau\Big[&\dot x^\mu(\tau)\dot x^\nu(\tau)+a^\mu(\tau)a^\nu(\tau)+b^\mu(\tau)c^\nu(\tau)+ \\ &+2\Big(\psi^\mu(\tau)\dot\psi^\nu(\tau)+\alpha^\mu(\tau)\alpha^\nu(\tau)+i\dot x^\mu(\tau)\psi^\nu(\tau)\psi(\tau)\cdot k\Big)\Big]\,{\rm e}^{ik\cdot x(\tau)}\end{aligned} \tag{2.33}$$

where α^μ are the additional bosonic ghosts arising from the nontrivial path integral measure for ψ^μ [35]. Their correlator is

$$\langle\alpha^\mu(\tau_1)\alpha^\nu(\tau_2)\rangle = \delta(\tau_1-\tau_2)\delta^{\mu\nu}\,. \tag{2.34}$$

In the perturbative expansion around flat space various worldline Feynman diagrams are linearly and logarithmically UV divergent. In fact the additional derivative couplings due to the metric worsen power counting in momentum space. The contributions from the measure, i.e. terms involving the ghost correlators, will always eliminate these divergences. Nevertheless, finite ambiguities are left over and dealt with by specifying a regularization scheme together with renormalization conditions.[3] These amount to the requirement that the path integral in (2.3) correspond precisely to the hamiltonian operator $H=(-\Box_A+m^2+\xi R)$ appearing in (2.2). These renormalization conditions produce a finite counterterm of the form

$$\Delta S_{CT} = \int_0^T d\tau\, 2\,V_{CT} \tag{2.35}$$

that must be added to the action (2.4). Three regularization schemes have been worked out in detail for this purpose: mode regularization (MR), time slicing (TS), and dimensional regularization (DR). The corresponding counterterms are given by

$$\begin{aligned}V_{MR} &= -\frac{1}{8}R+\frac{1}{8}g^{\mu\nu}\Gamma^\beta_{\mu\alpha}\Gamma^\alpha_{\nu\beta}\,,\\ V_{TS} &= -\frac{1}{8}R-\frac{1}{24}g^{\mu\nu}g^{\alpha\beta}g_{\lambda\rho}\Gamma^\lambda_{\mu\alpha}\Gamma^\rho_{\nu\beta}\,,\\ V_{DR} &= -\frac{1}{8}R\,.\end{aligned} \tag{2.36}$$

For details see [32, 31, 34].

In the present article, we will be interested in the graviton-photon amplitude in a constant electromagnetic background mediated by a virtual scalar particle loop. This means that we will only need to consider the linear coupling to the metric fluctuations around flat space. To this order the differences between the counterterms in (2.36) can

[3] We emphasize that these issues concern the one-dimensional worldline theory and bear no direct relation to the issue of regularization in spacetime, to be considered later.

be neglected since they are at least quadratic in metric fluctuations. On the other hand the leading part of the counterterm contained in the $-\frac{1}{8}R$ piece effectively changes the coupling $\xi \to \bar{\xi} = \xi - \frac{1}{4}$ in (2.4). This implies that to this order all three regularization schemes can be used interchangeably.

In the case of the spin 1/2 particle the coupling to R is fixed by the Dirac equation and corresponds to $\bar{\xi} = 0$. Additional details on the worldline formalism with background gravity can be found in [28, 35].

3. Calculation of the photon — graviton amplitude in a constant electromagnetic field: scalar loop

According to the above, the amplitude for the interaction of a graviton h and a photon A via a scalar loop, in the presence of a constant electromagnetic background field $F_{\mu\nu}$, is given by the following expression:

$$\begin{aligned}\langle h(k_1)A(k_2)\rangle &= (-ie)\Big(-\frac{\kappa}{4}\Big)\int_0^\infty \frac{dT}{T}\, e^{-m^2T}\int_P DxDaDbDc\, V^h_{\rm scal}[k_1,\varepsilon^h]V^A_{\rm scal}[k_2,\varepsilon^A]\times\\ &\quad\times\exp\Big[-\int_0^T d\tau\Big(\frac{1}{4}(\dot{x}^2+a^2+b\cdot c)+\frac{1}{2}ie\, x^\mu F_{\mu\nu}\dot{x}^\nu\Big)\Big]. \end{aligned}\tag{3.1}$$

Here $V^{A,h}_{\rm scal}$ represent the photon and graviton vertex operators for the scalar loop case, (2.5) and (2.31).

To start with, we perform the split (2.6) and the trivial x_0 integration which produces the global δ — function for momentum conservation. The remaining path integral $\int Dy(\tau)$ is gaussian and thus can be reduced to Wick contraction. Taking (2.17) into account, one obtains

$$\begin{aligned}\langle h(k_1)A(k_2)\rangle &= \frac{ie\kappa}{4}(2\pi)^D\delta(k_1+k_2)\int_0^\infty\frac{dT}{T}\, e^{-m^2T}(4\pi T)^{-\frac{D}{2}}{\rm det}^{-\frac{1}{2}}\Big[\frac{\sin(\mathcal{Z})}{\mathcal{Z}}\Big]\times\\ &\quad\times\Big\langle V^h_{\rm scal}[k_1,\varepsilon^h]V^A_{\rm scal}[k_2,\varepsilon^A]\Big\rangle\end{aligned}\tag{3.2}$$

($\mathcal{Z}_{\mu\nu} = eTF_{\mu\nu}$).

Performing the Wick contractions according to the rules (2.8), (2.32), and using (2.14), we obtain

$$\langle h(k_1)A(k_2)\rangle = (2\pi)^D\delta(k_1+k_2)\varepsilon^h_{\mu\nu}\varepsilon^A_\alpha\,\Pi^{\mu\nu,\alpha}_{\rm scal}(k)\tag{3.3}$$

where $k\equiv k_1$, and

$$\begin{aligned}\Pi^{\mu\nu,\alpha}_{\rm scal}(k) &= \frac{e\kappa}{4(4\pi)^{\frac{D}{2}}}\int_0^\infty\frac{dT}{T}\, e^{-m^2T}T^{-\frac{D}{2}}{\rm det}^{-\frac{1}{2}}\Big[\frac{\sin(\mathcal{Z})}{\mathcal{Z}}\Big]\times\\ &\quad\times\int_0^T d\tau_1\int_0^T d\tau_2\, {\rm e}^{-k\cdot\bar{\mathcal{G}}_{B12}\cdot k}I^{\mu\nu,\alpha}_{\rm scal}\,,\end{aligned}\tag{3.4}$$

$$\begin{aligned}I^{\mu\nu,\alpha}_{\rm scal} &= -(\ddot{\bar{\mathcal{G}}}{}^{\mu\nu}_{B11}-2\delta_{11}\delta^{\mu\nu})(k\cdot\dot{\bar{\mathcal{G}}}_{B12})^\alpha-\Big[\ddot{\bar{\mathcal{G}}}{}^{\mu\alpha}_{B12}\Big(\dot{\bar{\mathcal{G}}}_{B12}\cdot k\Big)^\nu+(\mu\leftrightarrow\nu)\Big]+\\ &\quad+\Big(\dot{\bar{\mathcal{G}}}_{B12}\cdot k\Big)^\mu\Big(\dot{\bar{\mathcal{G}}}_{B12}\cdot k\Big)^\nu\Big(k\cdot\dot{\bar{\mathcal{G}}}_{B12}\Big)^\alpha-4\bar{\xi}(\delta^{\mu\nu}k^2-k^\mu k^\nu)\Big(k\cdot\dot{\bar{\mathcal{G}}}_{B12}\Big)^\alpha.\end{aligned}\tag{3.5}$$

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It is useful to add to $I^{\mu\nu,\alpha}_{\rm scal}\,{\rm e}^{-k\cdot\bar{\mathcal{G}}_{B12}\cdot k}$ the total derivative term

$$\frac{1}{2}\frac{\partial}{\partial\tau_1}\Big[\bar{\dot{\mathcal{G}}}^{\mu\alpha}_{B12}\Big(\bar{\mathcal{G}}_{B12}\cdot k\Big)^{\nu}{\rm e}^{-k\cdot\bar{\mathcal{G}}_{B12}\cdot k}+(\mu\leftrightarrow\nu)\Big]\,. \tag{3.6}$$

Then $I^{\mu\nu,\alpha}_{\rm scal}$ gets replaced by $J^{\mu\nu,\alpha}_{\rm scal}$,

$$J^{\mu\nu,\alpha}_{\rm scal}=J^{\mu\nu,\alpha}_{\rm scal,1}+J^{(\mu\nu),\alpha}_{\rm scal,2}+J^{(\mu\nu),\alpha}_{\rm scal,3}+J^{\mu\nu,\alpha}_{\rm scal,4} \tag{3.7}$$

where $J^{(\mu\nu)}=\frac{1}{2}(J^{\mu\nu}+J^{\nu\mu})$ and

$$\begin{aligned}
J^{\mu\nu,\alpha}_{\rm scal,1}&=-\Big(\ddot{\mathcal{G}}^{\mu\nu}_{B11}-2\delta_{11}\delta^{\mu\nu}\Big)\Big(k\cdot\bar{\dot{\mathcal{G}}}_{B12}\Big)^{\alpha}\,,\\
J^{\mu\nu,\alpha}_{\rm scal,2}&=\bar{\dot{\mathcal{G}}}^{\mu\alpha}_{B12}\Big(\ddot{\mathcal{G}}_{B12}\cdot k\Big)^{\nu}-\ddot{\mathcal{G}}^{\nu\alpha}_{B12}\Big(\bar{\dot{\mathcal{G}}}_{B12}\cdot k\Big)^{\mu}\,,\\
J^{\mu\nu,\alpha}_{\rm scal,3}&=\Big(\bar{\dot{\mathcal{G}}}_{B12}\cdot k\Big)^{\mu}\Big[\Big(\bar{\dot{\mathcal{G}}}_{B12}\cdot k\Big)^{\nu}\Big(k\cdot\bar{\dot{\mathcal{G}}}_{B12}\Big)^{\alpha}-\bar{\dot{\mathcal{G}}}^{\nu\alpha}_{B12}k\cdot\dot{\mathcal{G}}_{B12}\cdot k\Big]\,,\\
J^{\mu\nu,\alpha}_{\rm scal,4}&=-4\bar{\xi}(\delta^{\mu\nu}k^2-k^{\mu}k^{\nu})\Big(k\cdot\bar{\dot{\mathcal{G}}}_{B12}\Big)^{\alpha}\,.
\end{aligned} \tag{3.8}$$

Before proceeding further, let us use this integral representation to analyse the general structure of this amplitude. Although our calculation is nonperturbative in the external field, we can, of course, use the series expansions of the worldline Green's functions (2.13) to compute the amplitude involving a given number of interactions with the field. It is then immediately seen that this amplitude is nonzero only if this number of interactions is odd, since otherwise the $\tau_{1,2}$ integrations vanish by antisymmetry. Also, the integrand contains terms which are singular at $T=0$, indicating UV divergences. While so far our calculation has been valid for any spacetime dimension D, the structure of these divergences depends on D. We therefore confine ourselves to the four dimensional case in the following. For $D=4$ the terms $J^{\mu\nu,\alpha}_{\rm scal,3,4}$ are UV finite, while $J^{\mu\nu,\alpha}_{\rm scal,1,2}$ contain terms with a logarithmic divergence at $T=0$. Those divergent terms involve the field only linearly, and are thus easy to compute by expanding the formulas (2.13) to the linear order in F. In dimensional regularisation, the result is

$$\Pi^{\mu\nu,\alpha}_{\rm scal,div}(k)=\frac{ie^2\kappa}{3(4\pi)^2}\frac{1}{D-4}C^{\mu\nu,\alpha} \tag{3.9}$$

where

$$C^{\mu\nu,\alpha}=(F\cdot k)^{\alpha}\delta^{\mu\nu}+F^{\mu\alpha}k^{\nu}+F^{\nu\alpha}k^{\mu}-(F\cdot k)^{\mu}\delta^{\nu\alpha}-(F\cdot k)^{\nu}\delta^{\mu\alpha}\,. \tag{3.10}$$

As expected, this counterterm is just the momentum space version ($f_{\mu\nu}=k_\mu\varepsilon_\nu-k_\nu\varepsilon_\mu$) of the tree level interaction term (1.2).

We perform renormalization by subtracting the amplitude at zero field strength and zero momentum. This can be done under the T - integral, leading to the following form of the renormalized amplitude $\bar{\Pi}$:

$$\begin{aligned}
\bar{\Pi}^{\mu\nu,\alpha}_{\rm scal}(k)&=\frac{e\kappa}{64\pi^2}\int_0^\infty\frac{dT}{T^3}\,{\rm e}^{-m^2T}\times\\
&\times\Big\{{\rm det}^{-\frac{1}{2}}\Big[\frac{\sin(\mathcal{Z})}{\mathcal{Z}}\Big]\int_0^T d\tau_1\int_0^T d\tau_2\,{\rm e}^{-k\cdot\bar{\mathcal{G}}_{B12}\cdot k}J^{\mu\nu,\alpha}_{\rm scal}+\frac{2}{3}iT^2eC^{\mu\nu,\alpha}\Big\}\,.
\end{aligned} \tag{3.11}$$

Further, again following [25, 36] we split $\mathcal{G}_{Bij}$ into

$$\mathcal{G}_B = \mathcal{S}_B + \mathcal{A}_B \tag{3.12}$$

where $\mathcal{S}_B^{\mu\nu}$ contains the even powers of $F^{\mu\nu}$ in the power series representation of $\mathcal{G}_B^{\mu\nu}$, and $\mathcal{A}_B^{\mu\nu}$ the odd ones. After this replacement all terms in the integrand are either symmetric or antisymmetric under the exchange $\tau_1 \leftrightarrow \tau_2$, and the antisymmetric ones can be deleted since their $\tau_{1,2}$ - integrals vanish. Note that the exponent $k \cdot \overline{\mathcal{G}}_{B12} \cdot k = k \cdot \overline{\mathcal{S}}_{B12} \cdot k$ is symmetric under this exchange.

Further, as usual [36] we rescale $\tau_i = Tu_i, i = 1, 2$, and use the translation invariance in τ to set $u_2 = 0$. After analytic continuation to Minkowski spacetime,[4] this leads to our final result for this amplitude:

$$\bar{\Pi}^{\mu\nu,\alpha}_{\rm scal}(k) = \frac{e\kappa}{64\pi^2}\int_0^\infty \frac{ds}{s}\,{\rm e}^{-ism^2}\left\{\det{}^{-\frac{1}{2}}\left[\frac{\sin(\mathcal{Z})}{\mathcal{Z}}\right]\int_0^1 du_1\,{\rm e}^{-k\cdot\overline{\mathcal{S}}_{B12}\cdot k}\sum_{m=1}^{4}\tilde{J}^{(\mu\nu),\alpha}_{\rm scal,m} + \frac{2}{3}ieC^{\mu\nu,\alpha}\right\} \tag{3.13}$$

where $C^{\mu\nu,\alpha}$ is as in (3.10) (with $\delta^{\mu\nu} \to \eta^{\mu\nu}$) and

$$\begin{aligned}
\tilde{J}^{\mu\nu,\alpha}_{\rm scal,1} &= -\Big(\ddot{\mathcal{S}}^{\mu\nu}_{B11} - 2\delta_{11}\eta^{\mu\nu}\Big)\Big(k\cdot\overline{\dot{\mathcal{A}}}_{B12}\Big)^\alpha,\\
\tilde{J}^{\mu\nu,\alpha}_{\rm scal,2} &= \dot{\mathcal{S}}^{\mu\alpha}_{B12}\Big(\ddot{\mathcal{A}}_{B12}\cdot k\Big)^\nu - \ddot{\mathcal{A}}^{\nu\alpha}_{B12}\Big(\dot{\mathcal{S}}_{B12}\cdot k\Big)^\mu + \overline{\dot{\mathcal{A}}}^{\mu\alpha}_{B12}\Big(\ddot{\mathcal{S}}_{B12}\cdot k\Big)^\nu - \ddot{\mathcal{S}}^{\nu\alpha}_{B12}\Big(\overline{\dot{\mathcal{A}}}_{B12}\cdot k\Big)^\mu,\\
\tilde{J}^{\mu\nu,\alpha}_{\rm scal,3} &= \Big(\dot{\mathcal{S}}_{B12}\cdot k\Big)^\mu\Big[\Big(\dot{\mathcal{S}}_{B12}\cdot k\Big)^\nu\Big(k\cdot\overline{\dot{\mathcal{A}}}_{B12}\Big)^\alpha +\\
&\qquad + \Big(\overline{\dot{\mathcal{A}}}_{B12}\cdot k\Big)^\nu\Big(k\cdot\dot{\mathcal{S}}_{B12}\Big)^\alpha - \overline{\dot{\mathcal{A}}}^{\nu\alpha}_{B12}k\cdot\dot{\mathcal{S}}_{B12}\cdot k\Big] +\\
&\quad +\Big(\overline{\dot{\mathcal{A}}}_{B12}\cdot k\Big)^\mu\Big[\Big(\dot{\mathcal{S}}_{B12}\cdot k\Big)^\nu\Big(k\cdot\dot{\mathcal{S}}_{B12}\Big)^\alpha +\\
&\qquad + \Big(\overline{\dot{\mathcal{A}}}_{B12}\cdot k\Big)^\nu\Big(k\cdot\overline{\dot{\mathcal{A}}}_{B12}\Big)^\alpha - \dot{\mathcal{S}}^{\nu\alpha}_{B12}k\cdot\dot{\mathcal{S}}_{B12}\cdot k\Big],\\
\tilde{J}^{\mu\nu,\alpha}_{\rm scal,4} &= -4\bar{\xi}(\eta^{\mu\nu}k^2 - k^\mu k^\nu)\Big(k\cdot\overline{\dot{\mathcal{A}}}_{B12}\Big)^\alpha.
\end{aligned} \tag{3.14}$$

For applications of this amplitude it will be necessary to write the matrix functions $\mathcal{S}$, $\mathcal{A}$ and their derivatives in more explicit form. A suitable representation has been given in [25, 36]. Let $\mathcal{F} = \frac{1}{2}(B^2 - E^2)$, $\mathcal{G} = \mathbf{B}\cdot\mathbf{E}$ the two Maxwell invariants and

$$a \equiv \sqrt{\sqrt{\mathcal{F}^2+\mathcal{G}^2}+\mathcal{F}}\,, \qquad b \equiv \sqrt{\sqrt{\mathcal{F}^2+\mathcal{G}^2}-\mathcal{F}} \tag{3.15}$$

so that $a^2 - b^2 = B^2 - E^2$ and $ab = \mathbf{E}\cdot\mathbf{B}$. Let

$$z_+ \equiv iesa\,, \qquad z_- \equiv -esb\,. \tag{3.16}$$

Then the determinant factor decomposes as

$$\det{}^{-\frac{1}{2}}\left[\frac{\sin(\mathcal{Z})}{\mathcal{Z}}\right] = \frac{z_+z_-}{\sinh(z_+)\sinh(z_-)}\,. \tag{3.17}$$

[4] We use the metric tensor $(\eta^{\mu\nu}) = {\rm diag}(-+++)$ in Minkowski spacetime. The analytic continuation from euclidean to Minkowski space amounts to substituting $\delta^{\mu\nu}\to\eta^{\mu\nu}$, $k^4\to -ik^0$, $T\to is$.

Let further

$$\hat{\mathcal{Z}}_+ \equiv \frac{aF - b\tilde{F}}{a^2+b^2}\,, \qquad \hat{\mathcal{Z}}_- \equiv -i\frac{bF + a\tilde{F}}{a^2+b^2}\,, \tag{3.18}$$

with $\tilde{F}_{\mu\nu} = \frac{1}{2}\varepsilon_{\mu\nu\alpha\beta}F^{\alpha\beta}$ the dual field strength tensor.[5] The matrices $\hat{\mathcal{Z}}_\pm$ fulfill

$$\hat{\mathcal{Z}}_+ \cdot \hat{\mathcal{Z}}_- = 0 \tag{3.19}$$

and

$$\hat{\mathcal{Z}}_+^2 = \frac{F^2 - b^2\mathbb{1}}{a^2+b^2}\,, \qquad \hat{\mathcal{Z}}_-^2 = -\frac{F^2 + a^2\mathbb{1}}{a^2+b^2}\,. \tag{3.20}$$

Then one has the following orthogonal matrix decompositions:

$$\begin{aligned}
\mathcal{S}_{B12} &= i\frac{s}{2}\sum_{a=\pm}\frac{A^a_{B12}}{z_a}\hat{\mathcal{Z}}_a^2\,,\\
\dot{\mathcal{S}}_{B12} &= -\sum_{a=\pm} S^a_{B12}\hat{\mathcal{Z}}_a^2\,,\\
\dot{\mathcal{A}}_{B12} &= -i\sum_{a=\pm} A^a_{B12}\hat{\mathcal{Z}}_a\,,\\
\ddot{\mathcal{S}}_{B12} &= \ddot{G}_{B12}\mathbb{1} - \frac{2i}{s}\sum_{a=\pm} z_a A^a_{B12}\hat{\mathcal{Z}}_a^2\,,\\
\ddot{\mathcal{A}}_{B12} &= \frac{2}{s}\sum_{a=\pm} z_a S^a_{B12}\hat{\mathcal{Z}}_a\,.
\end{aligned} \tag{3.21}$$

These formulas are written in terms of the following four basic scalar, dimensionless coefficient functions:

$$\begin{aligned}
S^\pm_{B12} &= \frac{\sinh(z_\pm\dot{G}_{B12})}{\sinh(z_\pm)}\,,\\
A^\pm_{B12} &= \frac{\cosh(z_\pm\dot{G}_{B12})}{\sinh(z_\pm)} - \frac{1}{z_\pm}\,.
\end{aligned} \tag{3.22}$$

Note that with respect to the exchange $u_1 \leftrightarrow u_2$ the $S^\pm_{B12}$ are odd and the $A^\pm_{B12}$ even. Thus only the latter have non-vanishing coincidence limits,

$$A^\pm_{Bii} = \coth(z_\pm) - \frac{1}{z_\pm}\,. \tag{3.23}$$

4. Calculation of the photon — graviton amplitude in a constant electromagnetic field: spinor loop

The corresponding calculation for the spinor loop case proceeds in a completely analogous way:

$$\langle h(k_1)A(k_2)\rangle = -\frac{1}{2}(-ie)\Big(-\frac{\kappa}{4}\Big)\int_0^\infty \frac{dT}{T}\,e^{-m^2T}\int_P DxDaDbDc\int_A D\psi D\alpha\,\times$$

[5] In our conventions $\varepsilon^{0123} = 1$.

$$
\begin{aligned}
&\times V^h_{\rm spin}[k_1,\varepsilon^h]V^A_{\rm spin}[k_2,\varepsilon^A]\exp\biggl[-\int_0^T d\tau\biggl(\frac{1}{4}(\dot x^2{+}a^2{+}b\cdot c)+\frac{1}{2}ie\,x^\mu F_{\mu\nu}\dot x^\nu+\\
&\qquad\qquad\qquad\qquad\qquad\qquad+\frac{1}{2}(\psi\cdot\dot\psi+\alpha^2)-ie\psi^\mu F_{\mu\nu}\psi^\nu\biggr)\biggr]\\
&=-\frac{ie\kappa}{8}2^{\frac{D}{2}}(2\pi)^D\delta(k_1+k_2)\int_0^\infty\frac{dT}{T}\,e^{-m^2T}(4\pi T)^{-\frac{D}{2}}\det{}^{-\frac{1}{2}}\biggl[\frac{\tan(\mathcal{Z})}{\mathcal{Z}}\biggr]\times\\
&\times\Bigl\langle V^h_{\rm spin}[k_1,\varepsilon^h]V^A_{\rm spin}[k_2,\varepsilon^A]\Bigr\rangle\,. \qquad (4.1)
\end{aligned}
$$

Here $V^{A,h}_{\rm spin}$ now represent the photon and graviton vertex operators for the fermion loop case, (2.25) and (2.33). The additional Wick contraction rules have been given in (2.20), (2.34). Performing the Wick contractions in (4.1) we obtain the analogue of (3.4),

$$
\begin{aligned}
\Pi^{\mu\nu,\alpha}_{\rm spin}(k)&=-\frac{e\kappa 2^{\frac{D}{2}}}{8(4\pi)^{\frac{D}{2}}}\int_0^\infty\frac{dT}{T}\,e^{-m^2T}T^{-\frac{D}{2}}\det{}^{-\frac{1}{2}}\biggl[\frac{\tan(\mathcal{Z})}{\mathcal{Z}}\biggr]\times\\
&\times\int_0^T d\tau_1\int_0^T d\tau_2\,\mathrm{e}^{-k\cdot\bar{\mathcal{G}}_{B12}\cdot k}I^{\mu\nu,\alpha}_{\rm spin}\,. \qquad (4.2)
\end{aligned}
$$

Here

$$
I^{\mu\nu,\alpha}_{\rm spin}=I^{\mu\nu,\alpha}_{\rm scal}(\bar\xi=0)+I^{(\mu\nu),\alpha}_{\rm extra} \qquad (4.3)
$$

where

$$
\begin{aligned}
I^{\mu\nu,\alpha}_{\rm extra}=&-\Bigl[\dot{\mathcal{G}}^{\mu\nu}_{F11}-2\delta_{11}\delta^{\mu\nu}+\Bigl(\mathcal{G}_{F11}\cdot k\Bigr)^\nu\Bigl(\bar{\dot{\mathcal{G}}}_{B12}\cdot k\Bigr)^\mu\Bigr]\Bigl(\mathcal{G}_{F22}\cdot k\Bigr)^\alpha+\\
&+\Bigl(\mathcal{G}_{F12}\cdot k\Bigr)^\mu\dot{\mathcal{G}}^{\nu\alpha}_{F12}-\mathcal{G}^{\mu\alpha}_{F12}\Bigl(\dot{\mathcal{G}}_{F12}\cdot k\Bigr)^\nu+\\
&+\Bigl[\mathcal{G}^{\nu\alpha}_{F12}\Bigl(k\cdot\mathcal{G}_{F12}\cdot k\Bigr)-\Bigl(\mathcal{G}_{F12}\cdot k\Bigr)^\nu\Bigl(k\cdot\mathcal{G}_{F12}\Bigr)^\alpha\Bigr]\Bigl(\bar{\dot{\mathcal{G}}}_{B12}\cdot k\Bigr)^\mu+\\
&+\Bigl(\dot{\mathcal{G}}^{\mu\nu}_{F11}-2\delta_{11}\delta^{\mu\nu}\Bigr)\Bigl(k\cdot\bar{\dot{\mathcal{G}}}_{B12}\Bigr)^\alpha+\\
&+\Bigl[\ddot{\mathcal{G}}^{\mu\nu}_{B11}-2\delta_{11}\delta^{\mu\nu}-\Bigl(\bar{\dot{\mathcal{G}}}_{B12}\cdot k\Bigr)^\mu\Bigl(\bar{\dot{\mathcal{G}}}_{B12}\cdot k\Bigr)^\nu\Bigr]\Bigl(\mathcal{G}_{F22}\cdot k\Bigr)^\alpha-\\
&-\Bigl[\ddot{\mathcal{G}}^{\mu\alpha}_{B12}-\Bigl(\bar{\dot{\mathcal{G}}}_{B12}\cdot k\Bigr)^\mu\Bigl(k\cdot\bar{\dot{\mathcal{G}}}_{B12}\Bigr)^\alpha\Bigr]\Bigl(\mathcal{G}_{F11}\cdot k\Bigr)^\nu\,. \qquad (4.4)
\end{aligned}
$$

It is useful to replace $I^{\mu\nu,\alpha}_{\rm spin}$ by

$$
J^{\mu\nu,\alpha}_{\rm spin}\equiv J^{\mu\nu,\alpha}_{\rm scal}(\bar\xi=0)+I^{(\mu\nu),\alpha}_{\rm extra}\,. \qquad (4.5)
$$

The integrand can then be rearranged in the following way (compare with (3.8) for the scalar loop):

$$
J^{\mu\nu,\alpha}_{\rm spin}=J^{\mu\nu,\alpha}_{\rm spin,1}+J^{(\mu\nu),\alpha}_{\rm spin,2}+J^{(\mu\nu),\alpha}_{\rm spin,3} \qquad (4.6)
$$

where

$$
\begin{aligned}
J^{\mu\nu,\alpha}_{\rm spin,1}&=\Bigl(\ddot{\mathcal{G}}^{\mu\nu}_{B11}-\dot{\mathcal{G}}^{\mu\nu}_{F11}\Bigr)\Bigl[(\bar{\dot{\mathcal{G}}}_{B21}+\mathcal{G}_{F22})\cdot k\Bigr]^\alpha\\
J^{\mu\nu,\alpha}_{\rm spin,2}&=\bar{\dot{\mathcal{G}}}^{\mu\alpha}_{B12}\Bigl(\ddot{\mathcal{G}}_{B12}\cdot k\Bigr)^\nu-\mathcal{G}^{\mu\alpha}_{F12}\Bigl(\dot{\mathcal{G}}_{F12}\cdot k\Bigr)^\nu-
\end{aligned}
$$

$$
\begin{aligned}
&\quad -\ddot{\mathcal{G}}^{\nu\alpha}_{B12}\Big[(\dot{\bar{\mathcal{G}}}_{B12}+\mathcal{G}_{F11})\cdot k\Big]^{\mu} + \dot{\mathcal{G}}^{\nu\alpha}_{F12}\Big(\mathcal{G}_{F12}\cdot k\Big)^{\mu} \\
J^{\mu\nu,\alpha}_{\text{spin},3} &= -\Big(\dot{\bar{\mathcal{G}}}_{B12}\cdot k\Big)^{\mu}\Big\{\Big[(\dot{\bar{\mathcal{G}}}_{B12}+\mathcal{G}_{F11})\cdot k\Big]^{\nu}\Big[(\dot{\bar{\mathcal{G}}}_{B21}+\mathcal{G}_{F22})\cdot k\Big]^{\alpha} - \\
&\quad -\Big(\mathcal{G}_{F12}k\Big)^{\nu}\Big(\mathcal{G}_{F21}k\Big)^{\alpha} + \overline{\ddot{\mathcal{G}}}^{\nu\alpha}_{B12}k\cdot\dot{\mathcal{G}}_{B12}\cdot k - \mathcal{G}^{\nu\alpha}_{F12}k\cdot\mathcal{G}_{F12}\cdot k\Big\}\,. \qquad (4.7)
\end{aligned}
$$

As in the scalar loop case, renormalization requires only the subtraction of a logarithmic divergence, yielding

$$
\begin{aligned}
\bar{\Pi}^{\mu\nu,\alpha}_{\text{spin}}(k) &= -\frac{e\kappa}{32\pi^2}\int_0^\infty \frac{dT}{T^3}\,\mathrm{e}^{-m^2T}\,\times \\
&\quad \times\Big\{\det{}^{-\frac{1}{2}}\Big[\frac{\tan(\mathcal{Z})}{\mathcal{Z}}\Big]\int_0^T d\tau_1\int_0^T d\tau_2\,\mathrm{e}^{-k\cdot\bar{\mathcal{G}}_{B12}\cdot k}J^{\mu\nu,\alpha}_{\text{spin}} - \frac{4}{3}iT^2eC^{\mu\nu,\alpha}\Big\}\,. \qquad (4.8)
\end{aligned}
$$

The tensor $C^{\mu\nu,\alpha}$ appearing in the counterterm is the same as in the scalar loop case, eq. (3.10).

We now need also the analogue of the decomposition formulas (3.12), (3.21), (3.22) for $\mathcal{G}_F$,

$$
\mathcal{G}_F = \mathcal{S}_F + \mathcal{A}_F\,, \qquad (4.9)
$$

$$
\begin{aligned}
\mathcal{S}_{F12} &= -\sum_{a=\pm} S^a_{F12}\,\hat{\mathcal{Z}}_a^2\,, \\
\mathcal{A}_{F12} &= -i\sum_{a=\pm} A^a_{F12}\,\hat{\mathcal{Z}}_a\,, \\
\dot{\mathcal{S}}_{F12} &= \dot{G}_{F12}\,\mathbb{1} - \frac{2i}{s}\sum_{a=\pm} z_a A^a_{F12}\,\hat{\mathcal{Z}}_a^2\,, \\
\dot{\mathcal{A}}_{F12} &= \frac{2}{s}\sum_{a=\pm} z_a S^a_{F12}\,\hat{\mathcal{Z}}_a\,, \qquad (4.10)
\end{aligned}
$$

written in terms of the basic coefficient functions

$$
\begin{aligned}
S^{\pm}_{F12} &\equiv G_{F12}\frac{\cosh(z_\pm\dot{G}_{B12})}{\cosh(z_\pm)}\,, \\
A^{\pm}_{F12} &\equiv G_{F12}\frac{\sinh(z_\pm\dot{G}_{B12})}{\cosh(z_\pm)}\,, \\
A^{\pm}_{F11} &= \tanh(z_\pm)\,.
\end{aligned}
$$

As in the bosonic case $S^{\pm}_{F12}$, ($A^{\pm}_{F12}$) are odd (even) with respect to $1 \leftrightarrow 2$.

Proceeding as in the scalar loop case, we obtain our final result for the amplitude:

$$
\begin{aligned}
\bar{\Pi}^{\mu\nu,\alpha}_{\text{spin}}(k) &= -\frac{e\kappa}{32\pi^2}\int_0^\infty \frac{ds}{s}\,\mathrm{e}^{-ism^2}\Big\{\frac{z_+z_-}{\tanh(z_+)\tanh(z_-)}\int_0^1 du_1\,\times \\
&\qquad \times\,\mathrm{e}^{-k\cdot\bar{\mathcal{S}}_{B12}\cdot k}\sum_{m=1}^{3}\tilde{J}^{(\mu\nu),\alpha}_{\text{spin,m}} - \frac{4}{3}ieC^{\mu\nu,\alpha}\Big\} \qquad (4.11)
\end{aligned}
$$

where

$$
\begin{aligned}
\tilde{J}^{\mu\nu,\alpha}_{\text{spin},1} &= -\Big(\ddot{\mathcal{S}}^{\mu\nu}_{B11} - \dot{\mathcal{S}}^{\mu\nu}_{F11}\Big)\Big(k\cdot(\overline{\dot{\mathcal{A}}}_{B12} + \mathcal{A}_{F22})\Big)^{\alpha}, \\
\tilde{J}^{\mu\nu,\alpha}_{\text{spin},2} &= \dot{\mathcal{S}}^{\mu\alpha}_{B12}\Big(\ddot{\mathcal{A}}_{B12}\cdot k\Big)^{\nu} - \mathcal{S}^{\mu\alpha}_{F12}\Big(\dot{\mathcal{A}}_{F12}\cdot k\Big)^{\nu} - \ddot{\mathcal{A}}^{\nu\alpha}_{B12}\Big(\dot{\mathcal{S}}_{B12}\cdot k\Big)^{\mu} + \\
&\quad + \dot{\mathcal{A}}^{\nu\alpha}_{F12}\Big(\mathcal{S}_{F12}\cdot k\Big)^{\mu} + \overline{\dot{\mathcal{A}}}^{\mu\alpha}_{B12}\Big(\ddot{\mathcal{S}}_{B12}\cdot k\Big)^{\nu} - \mathcal{A}^{\mu\alpha}_{F12}\Big(\dot{\mathcal{S}}_{F12}\cdot k\Big)^{\nu} - \\
&\quad - \ddot{\mathcal{S}}^{\nu\alpha}_{B12}\Big((\overline{\dot{\mathcal{A}}}_{B12} + \mathcal{A}_{F11})\cdot k\Big)^{\mu} + \dot{\mathcal{S}}^{\nu\alpha}_{F12}\Big(\mathcal{A}_{F12}\cdot k\Big)^{\mu}, \\
\tilde{J}^{\mu\nu,\alpha}_{\text{spin},3} &= \Big(\dot{\mathcal{S}}_{B12}\cdot k\Big)^{\mu}\Big[\Big(\dot{\mathcal{S}}_{B12}\cdot k\Big)^{\nu}\Big(k\cdot(\overline{\dot{\mathcal{A}}}_{B12} + \mathcal{A}_{F11})\Big)^{\alpha} - \Big(\mathcal{S}_{F12}\cdot k\Big)^{\nu}\Big(k\cdot\mathcal{A}_{F12}\Big)^{\alpha} + \\
&\qquad + \Big(\Big(\overline{\dot{\mathcal{A}}}_{B12} + \mathcal{A}_{F11}\Big)\cdot k\Big)^{\nu}\Big(k\cdot\dot{\mathcal{S}}_{B12}\Big)^{\alpha} - \Big(\mathcal{A}_{F12}\cdot k\Big)^{\nu}\Big(k\cdot\mathcal{S}_{F12}\Big)^{\alpha} - \\
&\qquad - \overline{\dot{\mathcal{A}}}^{\nu\alpha}_{B12}k\cdot\dot{\mathcal{S}}_{B12}\cdot k + \mathcal{A}^{\nu\alpha}_{F12}k\cdot\mathcal{S}_{F12}\cdot k\Big] + \\
&\quad + \Big(\overline{\dot{\mathcal{A}}}_{B12}\cdot k\Big)^{\mu}\Big[\Big(\dot{\mathcal{S}}_{B12}\cdot k\Big)^{\nu}\Big(k\cdot\dot{\mathcal{S}}_{B12}\Big)^{\alpha} - \Big(\mathcal{S}_{F12}\cdot k\Big)^{\nu}\Big(k\cdot\mathcal{S}_{F12}\Big)^{\alpha} \\
&\qquad + \Big((\overline{\dot{\mathcal{A}}}_{B12} + \mathcal{A}_{F11})\cdot k\Big)^{\nu}\Big(k\cdot(\overline{\dot{\mathcal{A}}}_{B12} + \mathcal{A}_{F11})\Big)^{\alpha} - \\
&\qquad - \Big(\mathcal{A}_{F12}\cdot k\Big)^{\nu}\Big(k\cdot\mathcal{A}_{F12}\Big)^{\alpha} - \dot{\mathcal{S}}^{\nu\alpha}_{B12}k\cdot\dot{\mathcal{S}}_{B12}\cdot k + \mathcal{S}^{\nu\alpha}_{F12}k\cdot\mathcal{S}_{F12}\cdot k\Big].
\end{aligned}
\tag{4.12}
$$

5. Conclusions

We have obtained compact explicit integral representations for the one-loop photon-graviton amplitudes involving a scalar or spinor loop and a constant electromagnetic field. The use of the constant field worldline formalism along the lines of [25] has allowed us to achieve this with modest calculational effort, and without having to specialize to a special Lorentz frame. As usual in this formalism the calculation for the spinor loop case has been an extension of the scalar loop one. Since the application of this formalism to gravitational backgrounds is still novel we have presented the calculation in some detail.

We have verified that our results for these amplitudes obey the gravitational and gauge Ward identities. As a further check, we have also shown that their low energy limits agree with the result of an effective action calculation [39].

In the upcoming second part of this paper we use our results for a numerical study of the photon-graviton conversion process in a magnetic field.

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A. Ward identities

In this appendix we derive the relevant Ward identities and verify that our graviton-photon amplitudes satisfy them. This provides a good check on the correctness of our calculations.

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There are two types of Ward identities: one that originates from gauge invariance and one that follows from reparametrization invariance (general coordinate invariance).

Gauge transformations are defined by

$$\delta_G A_\mu = \partial_\mu \lambda\,, \qquad \delta_G g_{\mu\nu} = 0 \tag{A.1}$$

with an arbitrary local parameter λ. Then gauge invariance of the effective action

$$\delta_G \Gamma[g, A] = 0 \tag{A.2}$$

implies that

$$\nabla_\mu \left(\frac{1}{\sqrt{g}} \frac{\delta\Gamma}{\delta A_\mu} \right) = 0\,. \tag{A.3}$$

Similarly, infinitesimal reparametrizations are given by

$$\delta_R A_\mu = \xi^\nu \partial_\nu A_\mu + \partial_\mu \xi^\nu A_\nu\,, \qquad \delta_R g_{\mu\nu} = \nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu \tag{A.4}$$

with arbitrary local parameters ξ^μ. The invariance of the effective action

$$\delta_R \Gamma[g, A] = 0 \tag{A.5}$$

now implies

$$\nabla_\mu \left(\frac{2}{\sqrt{g}} \frac{\delta\Gamma}{\delta g_{\mu\nu}} + \frac{1}{\sqrt{g}} \frac{\delta\Gamma}{\delta A_\mu} A^\nu \right) - \frac{1}{\sqrt{g}} \frac{\delta\Gamma}{\delta A_\mu} \nabla^\nu A_\mu = 0\,. \tag{A.6}$$

The Ward identities thus obtained can be combined and written more conveniently using standard tensor calculus as follows

$$\begin{aligned} \partial_\mu \frac{\delta\Gamma}{\delta A_\mu} &= 0\,, \\ 2\partial_\mu \frac{\delta\Gamma}{\delta g_{\mu\nu}} + \frac{\delta\Gamma}{\delta A_\mu} \partial_\mu A^\nu + \Gamma^\nu_{\mu\lambda} \left(2\frac{\delta\Gamma}{\delta g_{\mu\lambda}} + \frac{\delta\Gamma}{\delta A_\mu} A^\lambda \right) - \frac{\delta\Gamma}{\delta A_\mu} \nabla^\nu A_\mu &= 0\,. \end{aligned} \tag{A.7}$$

Now we are ready to consider the special case of the graviton-photon correlation function in flat space and in a constant electromagnetic background $F_{\mu\nu}$ described by the gauge potential $\bar{A}_\mu(x) = \frac{1}{2} x^\nu F_{\nu\mu}$

$$\Gamma^{\mu\nu,\alpha}_{(x,y)} \equiv \left. \frac{\delta^2 \Gamma}{\delta g_{\mu\nu}(x) \delta A_\alpha(y)} \right|_{g_{\mu\nu} = \delta_{\mu\nu},\, A_\alpha = \bar{A}_\alpha}\,. \tag{A.8}$$

Taking functional derivatives on the general Ward identities (A.7) to relate them to eq. (A.8) we obtain

$$\begin{aligned} \partial^{(y)}_\alpha \Gamma^{\mu\nu,\alpha}_{(x,y)} &= 0\,, \\ 2\partial^{(x)}_\mu \Gamma^{\mu\nu,\alpha}_{(x,y)} + \left. \frac{\delta^2\Gamma}{\delta A_\mu(x) \delta A_\alpha(y)} \right| F_\mu{}^\nu + \left. \frac{\delta\Gamma}{\delta A_\mu(x)} \right| (\delta^{\alpha\nu} \partial^{(x)}_\mu - \delta^\alpha_\mu \partial^\nu_{(x)}) \delta^D(x - y) &= 0\,. \end{aligned} \tag{A.9}$$

Now we Fourier transform these identities to momentum space

$$\int dx_1 \dots dx_n \, e^{ik_1x_1+\dots+ik_nx_n} \, \Gamma_{(x_1,\dots,x_n)} = (2\pi)^D \delta(k_1 + \dots + k_n) \Gamma_{(k_1,\dots,k_n)}$$

and from eqs. (A.9) we obtain

$$\begin{aligned} k_\alpha \Gamma^{\mu\nu,\alpha}_{(k,-k)} &= 0\,, \\ 2k_\mu \Gamma^{\mu\nu,\alpha}_{(k,-k)} + i\Gamma^{\mu,\alpha}_{(k,-k)} F_\mu{}^\nu + \Gamma^\mu_{(0)} (\delta^{\alpha\nu} k_\mu - \delta^\alpha_\mu k^\nu) &= 0\,. \end{aligned} \tag{A.10}$$

One may already notice that the term proportional to $\Gamma^\mu_{(0)}$ can be discarded, since it vanishes at zero momentum. Thus we see that the gravitational Ward identity relates the graviton-photon amplitude to the photon-photon amplitude. The latter has been calculated in the worldline formalism in [25, 36]. The two point functions used above are simply related to the vacuum polarizations computed in the main text and in [25, 36] as follows

$$\begin{aligned} \Gamma^{\mu,\alpha}_{(k,-k)} &= -\Pi^{\mu,\alpha}(k)\,, \\ \kappa\Gamma^{\mu\nu,\alpha}_{(k,-k)} &= -\Pi^{\mu\nu,\alpha}(k)\,. \end{aligned} \tag{A.11}$$

Thus the expected Ward identities are

$$k_\alpha \Pi^{\mu\nu,\alpha}(k) = 0 \tag{A.12}$$

and

$$k_\mu \Pi^{\mu\nu,\alpha}(k) = \frac{i}{2} \kappa F^\nu{}_\mu \Pi^{\mu\alpha}(k)\,. \tag{A.13}$$

Let us start with the amplitude due to a scalar loop. These Ward identities are most easily checked using the form (3.8). To verify the gauge Ward identity (A.12) note that $J^{\mu\nu,\alpha}_{\text{scal},2,3,4}$ vanish at the integrand level when contracted with k_α. This is not the case for $k_\alpha J^{\mu\nu,\alpha}_{\text{scal},1}$, but for this term the integral vanishes because $k \cdot \overline{\dot{\mathcal{G}}}_{B12} \cdot k$ is antisymmetric in $\tau_{1,2}$.

The verification of the gravitational Ward identity (A.13) requires a bit more work, since here only $k_\mu J^{\mu\nu,\alpha}_{\text{scal},4}$ drops out at the integrand level. We can simplify it by adding a suitable total derivative term:

$$\begin{aligned} k_\mu J^{\mu\nu,\alpha}_{\text{scal}} \, \mathrm{e}^{(\cdot)} \to{}& k_\mu J^{\mu\nu,\alpha}_{\text{scal}} \, \mathrm{e}^{(\cdot)} - \Big(\ddot{\mathcal{G}}^{\nu\alpha}_{B11} - 2\delta_{11}\delta^{\nu\alpha} \Big) \frac{\partial}{\partial\tau_1} \mathrm{e}^{(\cdot)} \\ &+ \frac{1}{2}\frac{\partial}{\partial\tau_1} \bigg\{ \Big[\Big(k \cdot \overline{\dot{\mathcal{G}}}_{B12} \Big)^\alpha \Big(\overline{\dot{\mathcal{G}}}_{B12} \cdot k \Big)^\nu - \overline{\dot{\mathcal{G}}}^{\nu\alpha}_{B12} k \cdot \dot{\mathcal{G}}_{B12} \cdot k \Big] \mathrm{e}^{(\cdot)} \bigg\} \\ ={}& \bigg\{ \Big(k \cdot \overline{\dot{\mathcal{G}}}_{B12} \Big)^\alpha \Big((\overline{\ddot{\mathcal{G}}}_{B12} + 2\delta_{11}\mathbb{1}) \cdot k \Big)^\nu - \Big(\overline{\ddot{\mathcal{G}}}_{B12} + 2\delta_{11}\mathbb{1} \Big)^{\nu\alpha} k \cdot \dot{\mathcal{G}}_{B12} \cdot k \bigg\} \mathrm{e}^{(\cdot)} \\ ={}& \bigg\{ \Big(k \cdot \overline{\dot{\mathcal{G}}}_{B12} \Big)^\alpha \Big((\overline{\ddot{\mathcal{G}}_{B12} - \ddot{G}_{B12}}) \cdot k \Big)^\nu - \Big(\overline{\ddot{\mathcal{G}}_{B12} - \ddot{G}_{B12}} \Big)^{\nu\alpha} k \cdot \dot{\mathcal{G}}_{B12} \cdot k \bigg\} \, \mathrm{e}^{(\cdot)}\,. \end{aligned} \tag{A.14}$$

Here in the last step we have used the fact that $\overline{\dot{\mathcal{G}}}_{B12}$ and $k \cdot \dot{\mathcal{G}}_{B12} \cdot k$ have vanishing coincident limits to replace $2\delta_{11}$ by $\ddot{G}_{B11} - \ddot{G}_{B12} = 2\delta_{11} - 2\delta_{12}$.

Next, let us write down the representation corresponding to (3.4) for the photon - photon amplitude in a constant field [25, 36]:

$$\Pi^{\mu\alpha}_{\rm scal}(k) = -\frac{e^2}{(4\pi)^{\frac{D}{2}}}\int_0^\infty \frac{dT}{T}\, e^{-m^2T}T^{-\frac{D}{2}}{\rm det}^{-\frac{1}{2}}\left[\frac{\sin(\mathcal{Z})}{\mathcal{Z}}\right]\int_0^T d\tau_1\int_0^T d\tau_2 {\rm e}^{-k\cdot\overline{\mathcal{G}}_{B12}\cdot k} I^{\mu\alpha}_{\rm scal} \tag{A.15}$$

where

$$I^{\mu\alpha}_{\rm scal} = \ddot{\overline{\mathcal{G}}}{}^{\mu\alpha}_{B12}\, k\cdot\dot{\mathcal{G}}_{B12}\cdot k - \left(\dot{\overline{\mathcal{G}}}_{B12}\cdot k\right)^\mu\left(k\cdot\dot{\overline{\mathcal{G}}}_{B12}\right)^\alpha . \tag{A.16}$$

Using (3.4), (A.14), and (A.15), the gravitational Ward identity (A.13) can now be easily verified using the matrix identity

$$\ddot{\mathcal{G}}_{B12} - \ddot{G}_{B12} = 2ieF\cdot\dot{\mathcal{G}}_{B12}\,. \tag{A.17}$$

Let us now consider the amplitude due to a fermion loop. The gauge Ward identity is again easily seen to be satisfied. To check the more subtle gravitational Ward identities we need the photon-photon amplitude in the constant electromagnetic background [25, 36]

$$\Pi^{\mu\alpha}_{\rm spin}(k) = \frac{e^2 2^{\frac{D}{2}}}{2(4\pi)^{\frac{D}{2}}}\int_0^\infty \frac{dT}{T}\, e^{-m^2T}T^{-\frac{D}{2}}{\rm det}^{-\frac{1}{2}}\left[\frac{\tan(\mathcal{Z})}{\mathcal{Z}}\right]\int_0^T d\tau_1\int_0^T d\tau_2 {\rm e}^{-k\cdot\overline{\mathcal{G}}_{B12}\cdot k} I^{\mu\alpha}_{\rm spin} \tag{A.18}$$

where

$$I^{\mu\alpha}_{\rm spin} = I^{\mu\alpha}_{\rm scal} + I^{\mu\alpha}_{\rm extra} \tag{A.19}$$

with $I^{\mu\alpha}_{\rm scal}$ given in (A.16) and

$$\begin{aligned} I^{\mu\alpha}_{\rm extra} = &-\mathcal{G}^{\mu\alpha}_{F12}\Big(k\cdot\mathcal{G}_{F12}\cdot k\Big) + \Big(\mathcal{G}_{F12}\cdot k\Big)^\mu\Big(k\cdot\mathcal{G}_{F12}\Big)^\alpha + \Big(\mathcal{G}_{F11}\cdot k\Big)^\mu\Big(\mathcal{G}_{F22}\cdot k\Big)^\alpha - \\ &-\Big(\mathcal{G}_{F11}\cdot k\Big)^\mu\Big(k\cdot\dot{\overline{\mathcal{G}}}_{B12}\Big)^\alpha + \Big(\dot{\overline{\mathcal{G}}}_{B12}\cdot k\Big)^\mu\Big(\mathcal{G}_{F22}\cdot k\Big)^\alpha . \end{aligned} \tag{A.20}$$

We essentially proved earlier that the scalar part satisfies the gravitational Ward identity. Thus we contract the remaining terms with k^μ and add a suitable total derivative

$$k_\mu I^{(\mu\nu),\alpha}_{\rm extra}\,{\rm e}^{(\cdot)} + \frac{\partial}{\partial\tau_1}T^{\nu\alpha} \tag{A.21}$$

with

$$\begin{aligned} T^{\nu\alpha} = \frac{1}{2}\Big[&\Big(\mathcal{G}_{F11}\cdot k\Big)^\nu\Big(\mathcal{G}_{F22}\cdot k\Big)^\alpha + \Big(\mathcal{G}_{F11}\cdot k\Big)^\nu\Big(k\cdot\dot{\overline{\mathcal{G}}}_{B12}\Big)^\alpha \\ &+\mathcal{G}^{\nu\alpha}_{F12}\Big(k\cdot\mathcal{G}_{F12}\cdot k\Big) - \Big(\mathcal{G}_{F12}\cdot k\Big)^\nu\Big(k\cdot\mathcal{G}_{F22}\Big)^\alpha - 2\Big(\dot{\overline{\mathcal{G}}}_{B12}\cdot k\Big)^\nu\Big(\mathcal{G}_{F22}\cdot k\Big)^\alpha\Big]{\rm e}^{(\cdot)} \end{aligned} \tag{A.22}$$

to obtain

$$\begin{aligned} k_\mu I^{(\mu\nu),\alpha}_{\rm extra}\,{\rm e}^{(\cdot)} + \frac{\partial}{\partial\tau_1}T^{\nu\alpha} = &-\Big[\Big(k\cdot\ddot{\overline{\mathcal{G}}}_{B12}\Big)^\nu + \Big(k\cdot\dot{\mathcal{G}}_{F11}\Big)^\nu\Big]\Big(\mathcal{G}_{F22}\cdot k\Big)^\alpha + \\ &+\dot{\mathcal{G}}^{\nu\alpha}_{F12}\Big(k\cdot\mathcal{G}_{F12}\cdot k\Big) - \Big(\dot{\mathcal{G}}_{F12}\cdot k\Big)^\nu\Big(k\cdot\mathcal{G}_{F12}\Big)^\alpha + \\ &+\Big[\Big(k\cdot\dot{\mathcal{G}}_{F11}\Big)^\nu - 2\delta_{11}k^\nu\Big]\Big(k\cdot\dot{\overline{\mathcal{G}}}_{B12}\Big)^\alpha . \end{aligned} \tag{A.23}$$

Now using

$$\ddot{\mathcal{G}}_{B12} = \ddot{G}_{B12} + 2ieF\dot{\mathcal{G}}_{B12}\,,$$
$$\dot{\mathcal{G}}_{F12} = 2\delta_{12} + 2ieF\mathcal{G}_{F12} \tag{A.24}$$

one can show that eq. (A.13) holds.

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